The Modelling and Control of Flywheel Inverted Pendulum System

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Abstract—Inverted pendulum is a control system, with the feature of high order, multi-variable, non-linearity and unstable naturally. It is quite important for us to study its balance and stability in control engineering field. In this paper, a flywheel inverted pendulum, as an object to be controlled, the dynamic model has been established, and the mathematical model which established has been certificated and did system performance analysis. Then, as the linear model, the fuzzy controller is designed based on packet control. The method can decrease the number of fuzzy rules and avoid the rule’s explosion. The simulation results show that the fuzzy controller makes the system achieve stabilization and balance control.

Keywords-flywheel inverted pendulum system; dynamic model; fuzzy control

I. INTRODUCTION

Inverted pendulum is a control system, with the feature of high order, multi-variable, non-linearity and unstable naturally. In a traditional, the model of inverted pendulum balance research which concentrate on pendulum fixed on a moving cart, by control the moving cart so that control the balance of pendulum[1][2]. [3] Advanced that biped robot step forward have been abstracted an inverted pendulum model which based on a turning flywheel, support leg have been abstracted a massless pendulum, and the pendulum that have been flexed, and body have been abstracted a flywheel. The mathematics model of inverted pendulum established by Newton-Euler approach, however, author’s analysis method very complex.

Up to mow, there are already many research methods for conventional control method of inverted pendulum, [4] used PID method, optimal control strategy LQR method and pole assignment method to design controllers, at the same time, these simulation effect have been studied by comparison. [5]established a T-S fuzzy control rules based on fuzzy model identification for cart-pendulum model of state space discretization, compare to directly control for cart-pendulum nonlinear model, the fuzzy rules much simple, and the control effect is rather better. [6] Used variable universe adaptive fuzzy controllers realized simulation experiment of four inverted pendulum. The results of simulation show that system is very well for stability and robustness, and by the way, it also shows that cart-pendulum moved to an appointed location.

However, up to now, it is very few for learn to flywheel inverted pendulum, and the method of research also much complex. [7] Takes a vertical downward pendulum with a flywheel stabled at vertical upward position finally by packet control method. Due to control difficulty is much larger, so the pendulum have been stabilized in±6.34°. [8] Used a method which said that neural network NARMA control of gyroscopic inverted pendulum to control flywheel-inverted pendulum (the paper said gyroscopic inverted pendulum), the results shows that the control method is effect for pendulum deviate vertical position±10°.

At present, it is fewer for the research about flywheel inverted pendulum in our country. Especially, its research of control method is at primary stage, need to further exploration.

In this paper, taking an inverted pendulum balance control system based on flywheel, the planar dynamic model have been established by Lagrange equation, then designed a fuzzy controller which based on packet control, in order to realized balance control of inverted pendulum based on flywheel.

II. DESCRIPTION AND MODELLING FOR AN INVERTED PENDULUM SYSTEM BASED ON FLYWHEEL

In this paper, the inverted pendulum system based on flywheel can be abstracted a flywheel with circular ring and a rod with uniform quality. Flywheel depends on motor driving formed reaction torque in vertical plan makes the pendulum finally stabled at vertical upward position.

As showed in Figure 1, rectangular coordinates system \( o-x'y' \) have been appeared in moving plane. Denote by \( m_1, o_1, (x_1, y_1), I_1, \theta \) and \( L_1 \) the pendulum mass, its center of mass, its position coordinates, its moment of inertia relative to the origin O, the angle between pendulum and vertical direction and the distance from the origin O to the centre of mass of pendulum, respectively. Denote by \( m_2, o_2, (x_2, y_2), I_2, \varphi \) and \( L_2 \) the flywheel mass, its center of mass, its position coordinates, its moment of inertia relative to the center of mass of flywheel, flywheel rotation angle and the distance from the flywheel centre of mass to the centre of mass of pendulum, respectively.

Taking Lagrange equation derived kinematics equation [9]:

\[
L(q, \dot{q}) = T(q, \dot{q}) - V(q, \dot{q})
\]

Here, \( L, T, V \) and \( q \) are respectively, the Lagrangian operator,
the kinetic energy of system, the potential energy of system and the generalized coordinates of system. Lagrange equation can be showed in:
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = f_i
\]
(2)

Where \(i=1, 2, 3, \ldots n\), \(f_i\) and \(D\) corresponding respectively to the external force which system by generalized coordinates and dissipative force function. Let \(\theta\) and \(\phi\) are system generalized coordinates.

The total kinetic energy of system:
\[
T = T_n + T_o
\]
(3)

Where \(T_n\) and \(T_o\) are respectively, the kinetic energy of the pendulum bar and the flywheel.

The kinetic energy of pendulum bar:
\[
T_n = \frac{1}{2} (m v_n^2 + I_n \dot{\theta}^2)
\]
(4)

The kinetic energy of flywheel:
\[
T_o = \frac{1}{2} (m v_o^2 + I_o \dot{\phi}^2)
\]
(5)

The total kinetic energy of system:
\[
T = \frac{1}{2} (m v_n^2 + I_n \dot{\theta}^2) + \frac{1}{2} (m v_o^2 + I_o \dot{\phi}^2)
\]
(6)

The potential of pendulum bar:
\[
V_n = m g L \cos \theta
\]
(7)

The potential of flywheel:
\[
V_o = m g (L_1 + L_2) \cos \theta
\]
(8)

Combined upside equations, let \(L_1 = L_2\), and obtained the Lagrangian operator:
\[
L = \frac{1}{2} (m v_n^2 + 4 m L_2^2 + I_1 \dot{\theta}^2 + I_2 \dot{\phi}^2) - (m + 2 m_c) g L \cos \theta
\]
(9)

The generalized moment of system:
\[
f = (-I_1 \ddot{\theta} - u - I_1 \dot{\theta}^2)
\]
(11)

Where \(u\) are driving moment of flywheel(that is output moment of reducer).

Dissipative force of pendulum bar:
\[
\tau_n = -c_n \dot{\theta}
\]

Dissipative force of flywheel:
\[
\tau_o = -c_o \dot{\phi}
\]

Where \(c_n\) and \(c_o\) are respectively, the friction factor of pendulum bar around \(O\) rotation and flywheel around \(\alpha_i\) rotation. Finally,

\[
(a + I_1) \ddot{\phi} + I_1 \dot{\phi} = b \sin \theta - c_\theta \dot{\theta}
\]
(12)

Where \(a = m L_2^2 + 4 m L_2^2 + I_1, b = (m_c + 2 m_c) g L\).

In condition of ignored air resistance, the upper is exact mathematical model expression of flywheel-inverted pendulum system.

III. THE LINEAR MODEL AND SYSTEM PERFORMANCE ANALYSIS OF INVERTED PENDULUM

In order to convenient analysis and computation, when \(\theta\) is smaller, according to taylor series expansion using approximate treatment, obtained:
\[
\begin{cases}
(a + I_1) \ddot{\theta} + I_1 \dot{\theta} = b \theta - c_\theta \dot{\theta} \\
I_1 \dot{\phi} = u - c_\phi \dot{\phi}
\end{cases}
\]
(13)

The upper is linearized mathematical model of inverted pendulum based on flywheel.

Selecting \(x = (\theta, \phi, \dot{\theta}, \dot{\phi})\) is state variable of system, here the state space expression of system:
\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi} \\
\theta \\
\phi
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-b & a & 0 & c_o \\
0 & 0 & 0 & 1 \\
-b & a & 0 & (a+c) / a
\end{bmatrix}
\begin{bmatrix}
\theta \\
\phi \\
\theta \end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
(14)

Here, \(c_o = 9.4 \times 10^{-3} Nm/sec\), \(c_\theta = 3.0 \times 10^{-3} Nm/sec\), \(L_1 = 0.25 m\), \(m_1 = 1.2 kg\), \(m_2 = 0.46 kg\), \(I_1 = 2.504 \times 10^{-3} kgm^2\), \(I_2 = 3.423 \times 10^{-3} kgm^2\), taking these parameters into expression(14), the state space expression of system changed into:
\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi} \\
\theta \\
\phi
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
24.158 & -0.044 & 0 & 0.0014 \\
0 & 0 & 0 & 1 \\
-24.158 & 0.044 & 0 & -0.089
\end{bmatrix}
\begin{bmatrix}
\theta \\
\phi \\
\theta \end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
296.667
\end{bmatrix}
\]
(15)

According to optimal control theory of linear quadratic form, in condition of state feedback control law, the system performance index can be reached the minimum. Characteristic equation of system \(det(\lambda I - A) = 0\), by calculating, the eigenvalue were \(\lambda_1 = 4.8924, \lambda_2 = 4.9378, \lambda_3 = -0.0896\). As one of eigenvalues is located in right half-plane of complex frequency domain, so, the system is unstable. The controllability matrix of system \(M = \left[ \begin{array}{c} B \\ AB \\ A'B \\ A''B \end{array} \right] \), \(\text{rank}(M) = 4\), therefore, the system is complex control.

IV. MODEL VERIFICATION

Above derivation process, we used a lot of approximate conditions, so the model whether is correct or not need to
verification. Here we used said necessary condition method, that means using MATLAB tools to simulation.

A. Verification of exact model

First, we will verify a model that before linearized. We were only ignored air resistance, even the angle of pendulum changed a lot, the model should be more exactly.

Using MATLAB/Simulink toolbox to verify the model, the model of system showed in Figure 2. According to actual situation, because of air resistance, the angle of pendulum should be do damping motion when input torque is zero and vertical upward declination of pendulum is 0.1rad, variation curve of pendulum should be change nearby 3.14rad, and less than 6.28rad, and stabled at vertical direction finally, that is 3.14rad. Figure 3 shows simulation curve which according to upper simulation block diagram.

According to Figure 3, as the time being, we can see that the declination of vertical pendulum gradually tend to 3.14rad, the phenomenon matched with actual situation, so the results shows that the exact model is correct.

B. Verification of model after linearization

Here we used exact model and model after literalized separately calculating step response of system (the initial value of pendulum angle is zero (vertical upward)).

According to Figure 4 and Figure 5, we can seen that both the model of exact and that of after linearization are nearly same about stepresponse, but the declination the more, the last the error is larger. However, both show that the system is unstable naturally when the initial value of pendulum angle is zero.

![Figure 2. The exact model of inverted pendulum under MATLAB/Simulink circumstance](image)

![Figure 3. The variable of pendulum angle with input moment is zero and vertical declination is 0.1rad](image)

![Figure 4. Response curve of pendulum angle when input torque of exact model is step](image)

![Figure 5. Response curve of pendulum angle when input torque of model after linearization is step](image)
V. DESIGN OF FUZZY CONTROLLER BASED ON PACKET CONTROL

From the equation (15), we can see that the system of inverted pendulum based on flywheel have four state variable, as to ensure that the exact of inverted pendulum system, the four state variable should be the system output and return to system input, therefore, the number of input variable of fuzzy controller would be four. If we designed fuzzy controller directly, must be meet the problem rule explosion, in order to avoid this problem and advance efficiency of controller. In this paper, we used fuzzy controller based on packet control.

For the four inputs available of fuzzy controller, the relation between declination and angular velocity of pendulum are very close, and the relation between declination and angular velocity of flywheel are very close too. However, the relationship between pendulum and flywheel are alleviation a little. Hence, the four input variable of fuzzy controller can be divided into two groups, that is pendulum declination and angular velocity are one group; flywheel rotation angle and angular velocity are another. As a result, the rules of control can be divided into two kinds. One is pendulum declination and angular velocity looked as condition, the other is flywheel rotation angle and angular velocity looked as condition, there conclusion both are control torque. Control torques of input both relationship to pendulum and flywheel. Thus, the size of control torque both consideration to pendulum declination and flywheel angular velocity two control targets, so that can be realized successfully for pendulum system[10].

A. Design of fuzzy controller

According to above discussion about fuzzy control rules, fuzzy controller would be use Mamdani class, there have four input variable, where $\theta$, $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ are respectively, the declination of pendulum, the angular velocity, the rotation angle of flywheel and angular velocity, output variable which is driving moment of flywheel. The universe are separately $[-3,3]$, $[-3,3]$, $[-2,2]$, $[-2,2]$, $[-3,3]$, using triangle, uniform distribution and total overlapping. The fuzzy subsets of pendulum declination, angular velocity, rotation angle of flywheel and angular velocity are $[NB \ NM \ NS \ Z \ PS \ PM \ PB]$, $[-3,3]$, $[-2,2]$, $[-3,3]$, $[NB \ NM \ NS \ Z \ PS \ PM \ PB]$, $[-3,3]$, $[-2,2]$, $[-3,3]$. Fuzzy rules tables are showed as table I and table II.

According to Figure 6, we can realize that the design of fuzzy controller based on packet control of inverted pendulum system. Where $k_{\theta}$, $k_{\dot{\theta}}$, $k_{\phi}$, $k_{\dot{\phi}}$ and $k_{f}$ are respectively, the quantization factor of pendulum declination, the quantization factor of angular velocity, quantization factor of rotation angle of flywheel, the quantization factor of rotation angular velocity of flywheel and the scale factor of input torque. By adjusting reasonably quantization factor and scale factor, it can be make fuzzy controller better.

VI. SIMULATION RESULTS

Let initial state of inverted pendulum based on flywheel is $X = [0.15rad, 0, 0, 0]^T$, that is declination of inverted pendulum is 0.15rad, other initial values of input are all zero. Used fuzzy controller which section V designed, and utilized MATLAB/Simulink toolbox that can be realized fuzzy control simulation of inverted pendulum based on flywheel, simulation results as showed in Figure 7. In these figures, black curve stand for variable of declination, red curve stand for variable of angular velocity.

From the simulation results, we can see that fuzzy controller which designed based on packet control can be effect for balance control of inverted pendulum based on flywheel in this paper. By 0.7s, the system makes pendulum stabled at vertical position, the response speed of whole course does quickly, and overshooting becomes less.

Robustness of the controller which designed in this paper would be verified as following, between 3s and 4s, step disturbance which amplitude is unit have been joined in system, response curve of system as showed in Figure 8. Simulation results shows that controller have better robustness, the system can be able to stable fast again after encountered disturbance.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>NB</th>
<th>NM</th>
<th>NS</th>
<th>Z</th>
<th>PS</th>
<th>PM</th>
<th>PB</th>
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<td>NM</td>
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<td>NM</td>
<td>NM</td>
<td>NS</td>
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<tr>
<td>$\dot{\theta}$</td>
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<td>NS</td>
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TABLE I. THE CONTROL RULE TABLE OF PENDULUM DECLINATION

<table>
<thead>
<tr>
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<th>Z</th>
<th>PM</th>
<th>PB</th>
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</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>NB</td>
<td>NB</td>
<td>NM</td>
<td>NM</td>
<td>Z</td>
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</table>

TABLE II. THE CONTROL RULE TABLE OF FLYWHEEL ROTATION

VII. CONCLUSION

According to an inverted pendulum based on flywheel, establishing a dynamics model, and analyzing its dynamics characteristics. In addition, we are designed a fuzzy controller that based on packet control, realized motion balance control of system. The method decreased the number of fuzzy rules, advanced the control quality of system. Simulation results show that fuzzy controller which our designed can control inverted pendulum based on flywheel effectively, realizing stable control of inverted
pendulum. The whole control process which is fast, smaller overshoot and have better disturbance rejection ability.

ACKNOWLEDGEMENT
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REFERENCES

Figure 1. The model of inverted pendulum system based on flywheel

Figure 6. Fuzzy control schematic diagram of flywheel-inverted pendulum system

Figure 7. Response curves of declination and angular velocity

Figure 8. In the condition of disturbance, response curve of declination and angular velocity