A Unified Approach to Optimal Proportional Navigation

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Abstract—A unified approach to optimal proportional navigation for missile is purposed based on guidance performance according to the case of attacking the target without any maneuverability. The mathematic model of guidance system is built according to nonlinear relative movement relationship between missile and target, based on zeroing relative distance and zeroing the rate of line of sight. Due to the model, the optimal true proportional navigation law of guidance system is derived by using optimizing the linear quadratic performance in the case of intercepting nonmaneuvering target, at the same time the unified optimal proportional guidance law is obtained according to optimizing the different linear quadratic performance. Finally, the target is intercepted applying the guidance law through the mathematic simulation, the performance of guidance system under two optimal proportional guidance law is compared on a common basis.

Keywords- proportional navigation; linear quadratic; optimal control; nonlinear

I. INTRODUCTION

Due to its simplicity of onboard implementation, proportional navigation (PN) has attracted a considerable amount of interest in the missile guidance literature since its inception in 1940s, especially PN law has been widely used for homing missile guidance. In its common form, the PN schemes can be categorized into two major classes, one is interceptor velocity referenced class, and another is the line-of-sight (LOS) referenced class. The latter is superior to the former in the aspect of mathematical tractability, and numerous papers have appeared in the literature dealing with the analytical study of the LOS referenced system such as the pure proportional navigation (PPN)\textsuperscript{[1]}, true proportional navigation (TPN)\textsuperscript{[2]}, ideal proportional navigation (IPN)\textsuperscript{[3]}, and biased proportional navigation (BPN)\textsuperscript{[4]}.

Although less success has been achieved in solving the nonlinear equations of interceptor velocity referenced system such as PPN and its variants, PPN is believed to be more practically implementable with requirement on forward acceleration/deceleration and with no constraints on the initial engagement. In [1] the closed-form solution of the equations of motion of an ideal missile pursuing a nonmaneuvering target according to the pure proportional navigation law is obtained as function of the polar coordinates for all navigation constants \( N \geq 2 \).

One of the challenges of investigating PN guidance laws is how to formulate a standard framework under which all PN schemes can be analyzed and solved in a unified manner. In [5] the major classes of proportional navigation, namely, TPN and PPN are analyzed and solved by a unified approach which defines the acceleration of the interceptor as being proportional to the rate of LOS angle with direction normal to an arbitrarily assigned vector. The other forms of proportional navigation laws also are analyzed by applying the same method in [6], and the capture region of ideal proportional navigation law is obtained in the three dimensions.

The optimal proportional guidance law is gained by utilizing optimal control theory when missile intercepting nonmaneuvering target. According to the philosophy of zeroing rate of LOS angle, the mathematic model between missile and target is built in [7], the optimal proportional guidance law is derived whose proportional coefficient is 3 based on the linear quadratic performance. However, the performance optimized only contains the maneuverable energy of missile’s acceleration. In additional, the other optimal proportional guidance law is obtained by zeroing the relative distance between missile and target in [8].

According to the investigation result of proportional guidance law, the authors address the problem of the optimal proportional law in a unified approach for homing missile to intercepting the target without any maneuverability in this article. Due to choosing the product of the relative distance and the rate of LOS angle as the state in the mathematic model is to be describing the relative relationship between missile and target. At the same time, a unified optimal proportional guidance law is proposed by using the linear quadratic performance including the maneuverable energy, the rate of LOS angle and the relative distance based on optimal control theory\textsuperscript{[9]}.

The paper is organized as follows: in Section 2 the mathematic model of the relative movement between missile and target is built. In Section 3 the unified optimal proportional guidance law based on optimal control theory is presented. Section 4 presents numerical simulation results and Section 5 summarizes the final conclusions.

II. MODEL OF ENGAGEMENT

The section presents the model of engagement for homing missiles to intercept nonmaneuvering targets.
The relative motion between a missile and a target in three dimensional spaces is depicted in Fig.1. In order to explain explicitly, the reference frame is introduced as following in terminal phase. The same to conventional configuration, target T is located above missile M. The distance between the target and missile is $R$, $OXYZ$ and $OX_4Y_4Z_4$ are the inertia coordinate frame and the line of sight (LOS) coordinate frame respectively in initial terminal phase, of which the origin $O$ is located in the gravity center of missile. The axis $OX_4$ of LOS coordinate frame is uniform to the line of sight whose direct points to target from missile, and the axis $OY_4$ perpendicular to the axis $OX_4$ is at the longitudinal plane and points to the top, he axis $OZ_4$ is perpendicular to the plane $OX_4Y_4$, which is determined by Cartesian frames.

It is assumed that the inertia coordinate frame becomes LOS coordinate frame after two transformations associated with rotating the angle $\theta$ around the axis $OY_4$ and then rotating the angle $q$ around the axis $OZ_4$. Therefore, $\dot{\theta}$ and $\dot{q}$ are the angular velocity of two rotation respectively. At the same time, missile is assumed as roll under the control of autopilot, thereby the relative motion model of three dimensional spaces is separated into two perpendicular channels and the guidance problem can be treated as a planar problem in each of those channel. In pitch channel the kinematical equation of relative motion in [7] is described by

$$R\ddot{q} = -2\dot{R}\dot{q} - a_u + a_r,$$  \hspace{1cm} (1)

where $a_u$ and $a_r$ are accelerations of missile and target in pitch channel, respectively.

Due to principle of designing guidance law for homing missile in the terminal phase, the traditional method zeroing the rate of of LOS angle is used to obtain the optimal proportional guidance law, however, the minimum relative distance, that is, the miss distance, is also expected to be obtained in intercepting nonmaneuvering target. In order to realize zeroing the rate of LOS angle and zeroing the relative distance, the new state $x = R\dot{q}$ is chosen as according to the kinematical equation (1), the kinematical equation of relative motion can be obtained as

$$\dot{x} = a(t)x - a_u + a_r,$$  \hspace{1cm} (2)

where $a(t) = -\dot{R}/R$.

Based on the kinematical equation (2), the unified optimal proportional guidance law is designed in following.

III. OPTIMAL PROPORTIONAL LAW DESIGN

The guidance problem for homing missiles to intercept nonmaneuvering target is considered a unified approach in terminal guidance phase. Based on zeroing the rate of LOS angle and zeroing the relative distance, the linear quadratic performance composed of the state of model (2) and the energy of missile’s acceleration is chosen as

$$J = \frac{1}{2}\int_0^\infty \left[ f^2(t) + \frac{1}{2} \int_0^t q^2(s) + r(t)u^2(s) \right] dt,$$  \hspace{1cm} (3)

where the parameter $f = \text{const}$, $q(t) \geq 0$ and $r(t) > 0$. According to the principle of designing terminal guidance law, $f$ is satisfied that $f \to \infty$, that is $x(t_f) = 0$, in order to obtain favourable guidance precision.

The optimal control law is obtained as following under the linear quadratic performance based on the optimal control theory[9]

$$u = r^{-1}(t)p(t)x,$$  \hspace{1cm} (4)

where the parameter $p(t)$ is the solution of the Riccati equation

$$\dot{p}(t) + 2a(t)p(t) - r^{-1}(t)p^2(t) + q(t) = 0.$$  \hspace{1cm} (5)

At the same time the condition is satisfied as $p(t_f) = c \to \infty$. In order to get the solution to the Riccati equation (5), it is assumed as $w(t) = p^{-1}(t)$, that is, $w(t)p(t) = 1$, and it is easy to obtain as

$$\dot{w}(t) = -r^{-1}(t)w(t)p(t).$$  \hspace{1cm} (6)

From (5), it is obtained as

$$\dot{w}(t) - 2a(t)w(t) - r^{-1}(t) + q(t)w^2(t) = 0$$

and the condition is satisfied as $w(t_f) = 0$.

The optimal proportional guidance laws is derived from the solution to the equation (6) based on the optimal control theory.

A. Traditional Optimal Proportional law

Due to the energy in the missile is limited, the energy of missile’s acceleration is firstly considered to optimize minimum in the terminal phase, and that is, the parameter...
\[ q(t) \text{ is chosen as } q(t) = 0 . \text{ Thus, it is obtained due to } (6) \]
\[ \dot{w}(t) - 2a(t)w(t) - r^{-1}(t) = 0 , \quad (7) \]

The analytical solution to (7) is obtained as
\[ w(t) = \exp \left[ 2a(t)dt \right] \left[ \int \exp \left( -2a(t)dt \right) r^{-1}(t)dt \right] . \]

From the parameter \( a(t) \), \( w(t) \) is obtained as
\[ w(t) = \frac{1}{R^2} \left[ \int \frac{R^2(t)}{R(t)} - R^2(t)dt \right] . \quad (8) \]

The parameter \( \dot{R}(t) \) is satisfied the condition \( \dot{R}(t) < 0 \) in the terminal guidance phase, so that the parameter \( r(t) \) is chosen as \( r(t) = -\dot{R}^{-1}(t) \).

From (8) and \( r(t) \) it is obtained
\[ w(t) = \left[ R^2(t) - R^2(t) \right] / 3R^2(t) . \quad (9) \]

The optimal proportional guidance law is obtained as
\[ u = \frac{3R^2(t)\dot{R}(t)}{R^2(t) - R^2(t)} x . \quad (10) \]

Zero miss distance is expected to be obtained, that is, \( R(t) = 0 \), at the same time \( x = R\dot{q} \), the guidance law (10) is obtained as
\[ u = 3\dot{R}(t)\dot{q}(t) . \quad (11) \]

The guidance law (11) is the same as the traditional optimal proportional guidance law with navigation coefficient 3 in [7].

**B. Unified Optimal Proportional Law**

The traditional optimal guidance law (11) is gained without considering the state in the linear quadratic performance. When \( q(t) > 0 \) is satisfied in performance, that is considering the state in the linear quadratic performance, the parameter \( q(t) \) is chosen as
\[ q(t) = -n(n + 3)\dot{R}(t) / R^2(t) , \quad (12) \]

where \( n > 0 \), and the parameter \( r(t) \) is chosen as \( r(t) = -\dot{R}^{-1}(t) \), the parameter \( p(t) \) is obtained as \( p(t) = (n + 3) / \dot{R}(t) \).

From (4) and \( x = R\dot{q} \), the unified optimal proportional guidance law is obtained as
\[ u = (n + 3)\dot{R}(t)\dot{q}(t) . \quad (13) \]

**Remark**

According to the optimal proportional guidance law (11) and (13), it is obtained the unified optimal guidance law for missile intercepting target without any maneuverability as following as
\[ u = N\dot{R}(t)\dot{q}(t) , \quad (14) \]

where navigation coefficient \( N \geq 3 \), based on zeroing the rate of LOS angle and zeroing the relative distance.

In additional, the traditional optimal proportional guidance law is obtained when navigation coefficient \( N = 3 \).

**IV. SIMULATION RESULT**

In this section, the feasibility and performance of the proposed unified optimal proportional guidance law is verified by the numerical simulations for homing missile against nonmaneuvering target in the terminal phase.

The simulation step is set to be 0.001s, the position coordinate of the target is \((5657m, 6000m, 5657m)\); the position coordinate of the missile is \((0m, 0m, 0m)\); the initial velocity of the target is \(V_T = 400 m/s\); the initial velocity of the missile is \(V_M = 1000 m/s\); the initial path angles of missile are 38 degree and 45 degree; the initial path angles of target are 0 degree and 180 degree. The navigation coefficients of the optimal proportional guidance law are chosen to 3 and 6 respectively in homing missile during the numerical simulations.

In addition, the acceleration of missile in the simulation is assumed to satisfy \( |a_n| \leq 8g \).

The simulation results are shown in Figures. Fig. 2 shows the curves of the rate of LOS angle \( \dot{q} \), Fig.3 shows the curves of the state \( R\dot{q} \), and Fig.4 shows the curves of \( a_n \) in pitch channel, when two optimal proportional guidance laws are used in the guidance system of missile. It is noted the real line represents the optimal proportional guidance law with navigation coefficient 3, and the dashed represents the optimal proportional guidance law with navigation coefficient 6.

It is observed from these figures that the energy of missile’s acceleration of guidance law with navigation coefficient 3 is less than that of guidance law with navigation coefficient 6. The rate of LOS angles hold zero nearby and the state are tend to zero under the control of two guidance law. However, The rate of LOS angle and the state of guidance law with navigation coefficient 6 is less than that of guidance law with navigation coefficient 3, mostly because the optimal proportional guidance law with navigation coefficient 6 is designed with considering the parameter \( q(t) = -18R(t) / R^2(t) \).

All of the simulation results testify to feasibility and performance of the proposed method.
Figure 2. The curves of $\dot{q}$.

Figure 3. The curves of state $\mathbf{R}\dot{q}$.

Figure 4. The curves of acceleration $a_{\text{pitch}}$.

V. CONCLUSION

The new state is introduced to describing the relative relationship between missile and target, hereby the kinematics model is built to realize the aim of zeroing the rate of LOS angle and zeroing the relative distance. According to linear quadratic performance based on the optimal control theory, the unified optimal proportional guidance law is proposed for homing missile to intercept the target without any maneuverability.

Simulation results have been presented which correspond to specified conditions. The results show that the unified optimal proportional guidance law has different performance according to different navigation coefficient through choosing the different parameter $q(t)$.

ACKNOWLEDGMENT

The valuable comments of the anonymous reviewers are gratefully appreciated.

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