**Computation of Volterra Kernels’ Identification to Riccati Nonlinear Equation**

Yunhai Wang, Jinglong Han  
Institute of Vibration Engineering Research  
Nanjing University of Aeronautics and Astronautics  
Nanjing 210016, Jiangsu, P. R. China  
E-mail: yantaicity@xyz.com  

Tingting Zhang  
College of Physical Science and Technology  
Sichuan University  
Chengdu, 610064, P. R. China  
E-mail:s091385@yahoo.cn

**Abstract**—In this paper, we will use a method of analytical calculation to identify a class weakly nonlinear dynamic system. Based on Volterra theory, the aim is to identify its first-order and second-order Volterra kernels. A model of such kind of nonlinear systems described by Riccati equation will be used to illustrate this method below.

**Keywords**—System Identification; Volterra Theory

I. INTRODUCTION

The Spanish mathematician Vito Volterra first introduced the notion of what is now known as a Volterra series in his Theory of Functionals. Volterra series have found a great deal of use in calculating small, but nevertheless troublesome, distortion terms in transistor amplifiers and other systems.

The Volterra theory was developed in 1930. The theory is based on functionals, or functions of other functions, and subsequently became a generation of the linear convolution integral approach that is applied to linear, time-invariant (LTI) system.

For many of the problems, the governing nonlinear equations are time invariant. In the field of aircraft, model identification by applying to Volterra theory has been studied by several authors. We refer the readers to 

II. VOLterra SERIES

The basic premise of the Volterra theory of nonlinear system is that any nonlinear system can be modeled as an infinite sum of multidimensional convolution integrals of increasing order. In continuous-time form, it has the form

\[ y(t) = h_0 + \int h_1(t-\tau)u(\tau)d\tau + \int \int h_2(t-\tau_1, t-\tau_2)u(\tau_1)u(\tau_2)d\tau_1 d\tau_2 + \cdots + \int \cdots \int h_n(t-\tau_1, \cdots, t-\tau_n)u(\tau_1)\cdots u(\tau_n)d\tau_1 \cdots d\tau_n + \cdots \]

\[ y(t) = h_0 + \int h_1(t-\tau)d\tau + \int \int h_2(\tau_1, \tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2 + \cdots + \int \cdots \int h_n(\tau_1, \cdots, \tau_n)u(t-\tau_1)\cdots u(t-\tau_n)d\tau_1 \cdots d\tau_n + \cdots \]

The value of \( h_0 \) is known based on the steady-state value of the system. It doesn’t need any special identification technique. The value of \( h_1(t) \) stands for the unit impulse response.

The higher order kernels \( h_1(t_1, t_2), \cdots, h_n(t_1, \cdots, t_n) \) are the responses of the nonlinear system to multiple unit impulses: e.g., \( h_2(t_1, t_2) \) is the response of the nonlinear system to two unit impulses applied at two points in time \( t_1 \) and \( t_2 \).

In this paper, we only discuss a class weakly nonlinear system. We assume that 1) the kernels, input function and output function are real-valued functions; 2) the system is causal and 3) the system is time invariant. Then supposed that \( h_0 = 0 \), there is no lose of generality, we now model this kind of system as a truncated, two sum of multidimensional convolution integrals of increasing order. The representation is rewritten below:
\[ y(t) = h_0 + \int_0^t h_1(t-\tau)u(\tau)d\tau + \int_0^t \int_0^t h_2(t-\tau_1,t-\tau_2)u(\tau_1)u(\tau_2)d\tau_1d\tau_2 \]

III. APPROACH TO IDENTIFICATION OF THE KERNELS

We note:

\[ y_{01}(s_1,t) = h_1(t-s_1) + h_2(t-s_1,t-s_1) \]

\[ y_{02}(s_1,t) = 2h_1(t-s_1) + 4h_2(t-s_1,t-s_1) \]

where \( y_{01}(s_1,t) \) is the response of the nonlinear system to a single unit impulse applied at time \( s_1 \); \( y_{02}(s_1,t) \) is the response of the nonlinear system to a single impulse at time \( s_1 \) but with double the amplitude.

Hence, we have:

\[ h_1(t-s_1) = 2y_{01}(s_1,t) - 0.5y_{02}(s_1,t) \]

Note:

\[ y_{11}(s_1,s_2,t) = y_{01}(s_1,s_2,t) + h_1(t-s_2) + h_2(t-s_2,t-s_1) + h_2(t-s_2,t-s_2) + 2h_2(t-s_1,t-s_1) \]

where \( y_{11}(s_1,s_2,t) \) is the response of the nonlinear system to two unit impulses, one at time \( s_1 \) and one at time \( s_2 \).

For a TI system, \( y_{01}(s_1,t) \) is just \( y_{02}(s_1,t) \) shifted in time to \( s_2 \).

Consequently, we have:

\[ h_2(t-s_1,t-s_2) = 0.5[y_{11}(s_1,s_2,t) - y_{01}(s_1,t) - y_{01}(s_2,t)] \]

A subsequent example will be given to illustrate this method.

IV. A MODEL DESCRIBED BY RICCATI EQUATION

Suppose there is a simple nonlinear system, for example, a series circuit. Varying input signal information, by gathering the corresponding output data at time \( t \), we can compute this system corresponding first-order and second-order kernels.

This example was ever studied by W. A. Silva[6] in a numerical method in 1999. In order to obtain output data instead of experiment, we need know this weakly nonlinear system's governing equation and solve it in an analytical method. The merit of using this way is easier to show the whole identification process in details.

The governing equation to the Riccati equation is:

\[ \frac{di(t)}{dt} + i(t) + 0.0001 \cdot i(t)^2 = v(t) \]

where \( i(t) \) is the current around the circuit and \( v(t) \) the input voltage. We will use a time step of 0.01 as the least interval of all the unit excitations to the nonlinear system.

As a first step, we have to get the value of \( y_{00}(s_1,t) \).

In order to do it, implementing a unit voltage excitation to this system at time 0, we solve its corresponding governing equation:

\[ \frac{di(t)}{dt} + i(t) + 0.0001 \cdot i(t)^2 = 0 \]

and find an analytical solution in the form as follows:

\[ y_{01}(0,t) = \frac{1}{10000} \cdot \frac{i_0 \cdot \exp(-t) + 0.0001}{1 + i_0 \cdot \exp(-t)}, t > 0. \]

In order to determine the const value of \( i_0 \), we recall a fact: \( \exp(0^+) = 0.01 \). So, we get \( i_0 = 0.01/10000 \).

As a second step, we have to get the value of \( y_{02}(s_1,t) \).

In order to do it, implementing a two-unit voltage excitation to this system at time 0, we solve its corresponding governing equation:

\[ \frac{di(t)}{dt} + i(t) + 0.0001 \cdot i(t)^2 = 0 \]

In the same way, we find its analytical solution in the form as follows:

\[ y_{02}(0,t) = \frac{10000 \cdot i_0 \cdot \exp(-t)}{1 + i_0 \cdot i_0 \cdot \exp(-t)}, t > 0. \]

where \( i_0 = 0.02/10000 \).

Hence, we can compute the first-order kernel of this system as follows:

\[ h_1(t) = 2y_{01}(0) - (1/2)y_{02}(0). \]

Now, we begin to compute the value of the second-order kernel at time \( t \).

Let \( s_1 = 0 \), \( s_2 \) varies. For example, we set: \( s_1 = 0 \) and \( s_2 = 3 \).

We have obtained:

\[ y_{01}(0,3) = \frac{10000 \cdot i_0 \cdot \exp(-3)}{1 + i_0 \cdot i_0 \cdot \exp(-3)} \]

\[ y_{02}(3,t) = \frac{10000 \cdot i_0 \cdot \exp(3-t)}{1 + i_0 \cdot i_0 \cdot \exp(3-t), t > 0.} \]

Supposed that two unit voltages as input information work together at time \( s_1 = 0 \) and \( s_2 = 3 \), respectively.

The solution of the Riccati equation is given below:

\[ y_{11}(0,3,t) = \frac{10000 \cdot i_0 \cdot \exp(3-t)}{1 + i_0 \cdot i_0 \cdot \exp(3-t), t > 3} \]

where

\[ i_0 = \frac{[10000 \cdot i_0 + y_{01}(0,3)]}{10000 \cdot i_0} = 0.01/10000. \]

Finally, we can compute the value of the second-order kernel function of the nonlinear system.

\[ h_2(t, t-3) = (1/2)[y_{11}(0,3,t) - y_{01}(0,t) - y_{01}(3,t)] \]
V. CONCLUSIONS

In this paper, we demonstrate an example in an analytical method to identify a class weakly nonlinear system's Volterra kernels. This method is clear and easier to help us understand the process of identification than numerical method. As can been seen, the component of the second-order kernel (Figure 2 or 3) is very much smaller than the first-order kernel (Figure 1). The first order kernel is sufficient to capture the response of this system.

The advantage of the Volterra series approach for modeling nonlinear system is that once the kernels are identified, the response of the nonlinear system to an arbitrary input can be predicted. If the system is not weakly nonlinear, the second-order kernel can provide an indication of the additive effect of the second-order nonlinearity with respective to the first-order term. Sometimes, we might have to identify the three or higher order Volterra kernels depending on the nonlinearity of the system of interest.

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REFERENCES


